Basic Concepts of compressible Fluid Flow

1.1 Velocity of sound: سرعة الصوت

For flows at low velocities, the velocity and pressure variations are small resulting in small and nigligable variation in fluid density such that the flow is considered incompressible one. While, for flows at high velocities, the variation in fluid velocity, pressure, temperature and density are large, and the flow is considered compressible.

The variation in density is a result of pressure changes in the flow field. The pressure is closely related to the velocity propagation of small pressure disturbance which is known as the velocity of sound "**c**"

for perfect gas, $P/\rho = R \cdot T$, then the velocity of sound is given by:

$$C = \sqrt{\gamma \cdot \frac{P}{\rho}} = \sqrt{\gamma \cdot R \cdot T} = \sqrt{\gamma \frac{\bar{R}}{\bar{M}} \cdot T}$$

where, γ = the specific heat ratio

R = gas constant (287 J/kg.k) هواء

$\bar{M}$ = molecular weight (8.314 kJ/mol.k)

$\bar{R}$ = the universal gas constant.

Table (1.1)

Typical values of the velocity of sound "c" of some common gases at 0°C

Gas	$\bar{M}$	γ	C (m/s)	Gas	$\bar{M}$	γ	C (m/s)
Air	28.96	1.404	331	Helium(He)	4.003	1.667	970
Argon (Ar)	39.94	1.667	308	H_2	2.016	1.407	1270
CO_2	44.01	1.3	258	O_2	32	1.4	315.1
Freon 12	120.9	1.139	146	Xenon (Xe)	131.3	1.667	170

1.2 Mach Number "M"

Mach number "M" is a dimensionless parameter which, is defined as the ratio between the velocity of a fluid or an object "v" and the velocity of sound "c"

$$M = \frac{V}{C}$$ سرعة الغاز / سرعة الصوت في نفس الغاز

The compressible flows are classified as following according Mach number M, as:

* Subsonic flow, for $M < 1$
* Sonic flow, for $M = 1$
* Supersonic flow, for $M > 1$
* Hypersonic flow, for $M > 5$

1.3 Reference Speeds

In incompressible fluid flow analysis, some characteristic speeds are considered as a reference values. The are stagnation speed of sound, maximum velocity and critical speed of sound.

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* Stagnation speed of sound "C_o"

Stagnation conditions is the condition of the flow at zero velocity, and denoted by 'o' subscript. Stagnation speed of sound is the sound speed at stagnation condition, i.e. $V=0$ and T_o. Therefore, it is defined as:

$$C_o = \sqrt{\gamma \cdot R \cdot T_o}$$

* Maximum speed "V_{max}"

It is defined as the maximum possible attained speed by flow that it can be achieved at $T=0$

$$V_{max} = \sqrt{\frac{2\gamma}{\gamma-1} \cdot R \cdot T_o}$$

* Dimensionless speed of the flow "M^*"

It is useful to normalized flow velocity with reference speed of constant value. This is the critical speed of sound

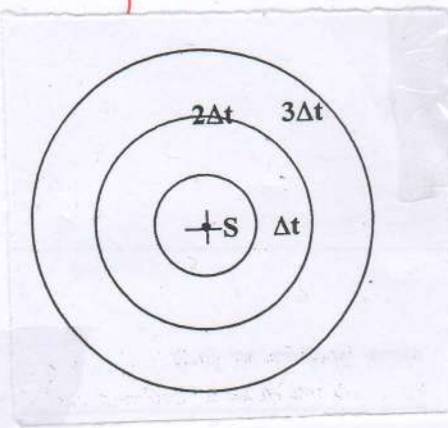
$$C^* = \sqrt{\gamma R T^*}$$

$$M^* = V / C^* = \frac{V}{\sqrt{\gamma R T^*}}$$

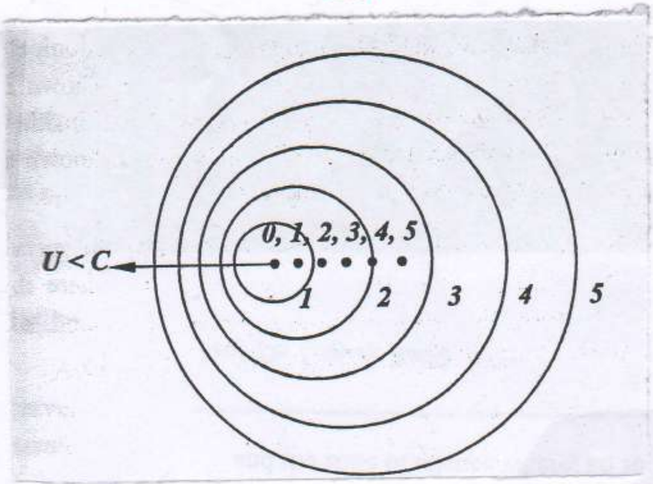
1.4 Pressure disturbance in a compressible fluid

when an object is moving through a compressible fluid, a pressure disturbance is generated. The pressure wave travels away of the disturbance source in all directions through the fluid with a velocity of sound "c" relative to the fluid.

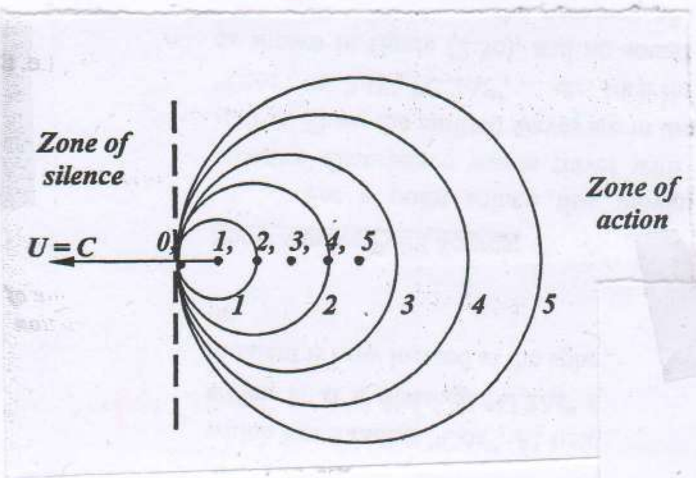
* Stationary Point Source



* Subsonic moving Point Source



* Sonic moving Point Source



* Super sonic moving point source

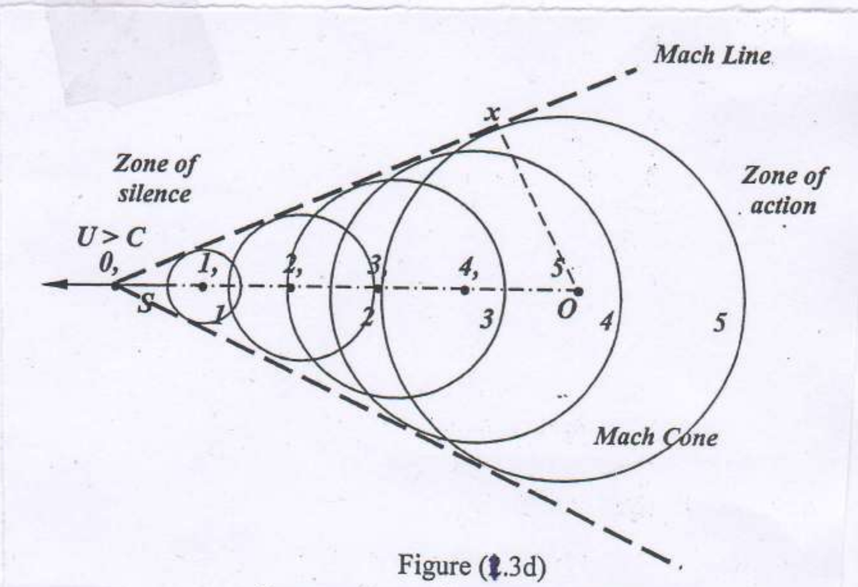


Figure (1.3d)
Supersonic moving point source U>C

Introduction:

In one-dimensional flow approximation, the variations of fluid flow properties are in flow direction only, and are uniform over any cross section. It means that the rate change of fluid properties is apprecciable only in streamwise direction and nigligible in the direction normal to the streamline. One-dimensional formulation gives a simple, rapid, and accurate calculation method for a great variety of practical engineering applications.

2-1 Governing Equations:

Consider the steady one dimensional isentropic flow through a passage of varying cross sectional area as shown in figure (2.1a). Also enthalpy-entropy (h-s) diagram for the flow considered is given in figure (2.1 b). The application of the physical governing equations to the control volume is given as follows.

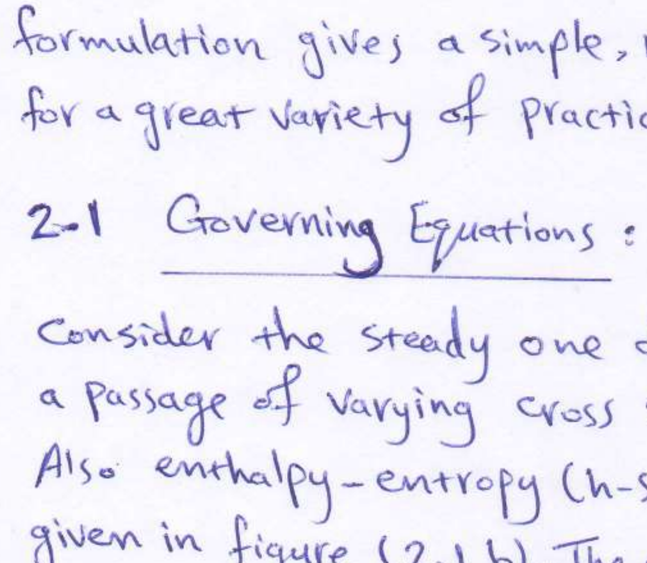


Figure (2.1 a)
control volume for simple area change flow

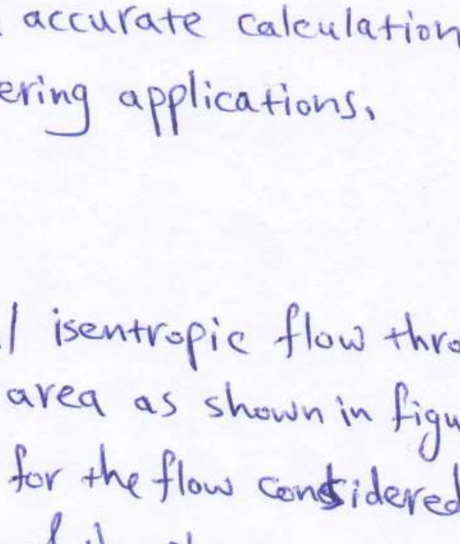


Figure (2.1 b)
h-s diagram for Isentropic flow

* Continuity equation:

for steady open flow system, the mass flow rate at any section is constant and is given by:

$$\dot{m} = \rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

* Second law of thermodynamics:

as the flow is adiabatic and reversible process, then

$$ds = \frac{dq}{T} = 0$$

$$s_1 = s_2$$

2.2 Isentropic Flow of a Perfect Gas:

Knowing that the equation of state for perfect gas, is given as:

$$P = \rho R T$$

Isentropic process relation for perfect gas flow is given as,

$$P \cdot \rho^{-\gamma} = \text{Constant} \quad \text{Then:}$$

$$\boxed{\frac{P_2}{P_1} = \left[\frac{\rho_2}{\rho_1}\right]^{\gamma} = \left[\frac{T_2}{T_1}\right]^{\frac{\gamma}{\gamma-1}}}$$

$$\boxed{\frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}} \quad , \quad \boxed{\frac{P_2}{P_1} = \left[\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\right]^{\frac{\gamma}{\gamma-1}}}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \left[\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\right]^{\frac{1}{\gamma-1}}}$$

Ex (2.1) A gas of molecular weight 4 and specific heat ratio $\gamma = 1.67$ flows through a variable area duct. At some point in the flow, the velocity is 250 m/s, the pressure is 100 kPa and temperature is 25°C. Find the Mach number at this point. At some other point in the flow, the temperature is −5 °C. Find Mach number, velocity, pressure at this point, assuming the flow is isentropic.

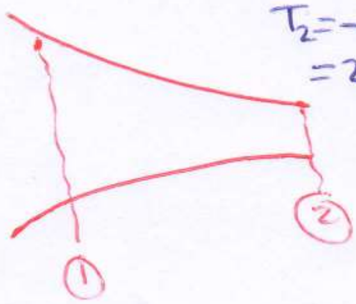
Solu:

$$R = \frac{\bar{R}}{\bar{M}} = \frac{8314}{4} = 2078.5 \text{ J/Kg.K}$$

$T_1 = 25 + 273 = 298$ K

$U_1 = 250 \frac{m}{s}$

$P_1 = 100$ kpa



$T_2 = -5 + 273 = 268$ K

$$C_1 = \sqrt{\gamma R T_1} = \sqrt{1.67 * 2078.5 * 298} = 1017 \text{ m/s}$$

$$M_1 = \frac{U_1}{C_1} = \frac{250}{1017} = 0.246$$

at section 2, the flow from 1−2 is isentropic

$$\Rightarrow \frac{T_1}{T_2} = \frac{298}{268} = \frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2} \Rightarrow M_2 = 0.634$$

$$C_2 = \sqrt{\gamma R T_2} = 964.5 \text{ m/s}$$

Gas speed at section 2 (U_2)

$$U_2 = M_2 * C_2 = 0.634 * 964.5 \Rightarrow U_2 = 611.5 \text{ m/s}$$

Pressure at section 2 (P_2)

$$P_2 = P_1 \left[\frac{1 + \frac{\gamma-1}{2} \cdot M_1^2}{1 + \frac{\gamma-1}{2} \cdot M_2^2}\right]^{\frac{\gamma}{\gamma-1}} = 100 \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = 100 \left(\frac{268}{298}\right)^{\frac{1.67}{0.67}}$$

$$\Rightarrow P_2 = 76.76 \text{ kpa}$$

2-3 Stagnation Conditions:

Stagnation conditions are the conditions of a reference state. It is defined as the state at which the velocity of the flow is zero.

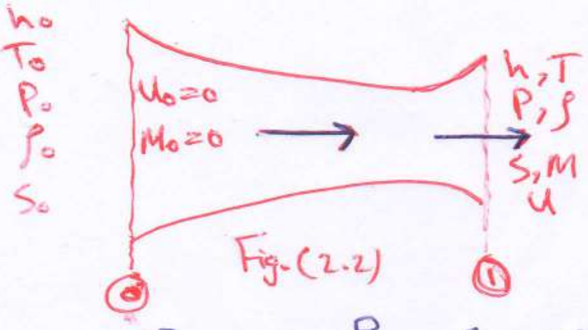
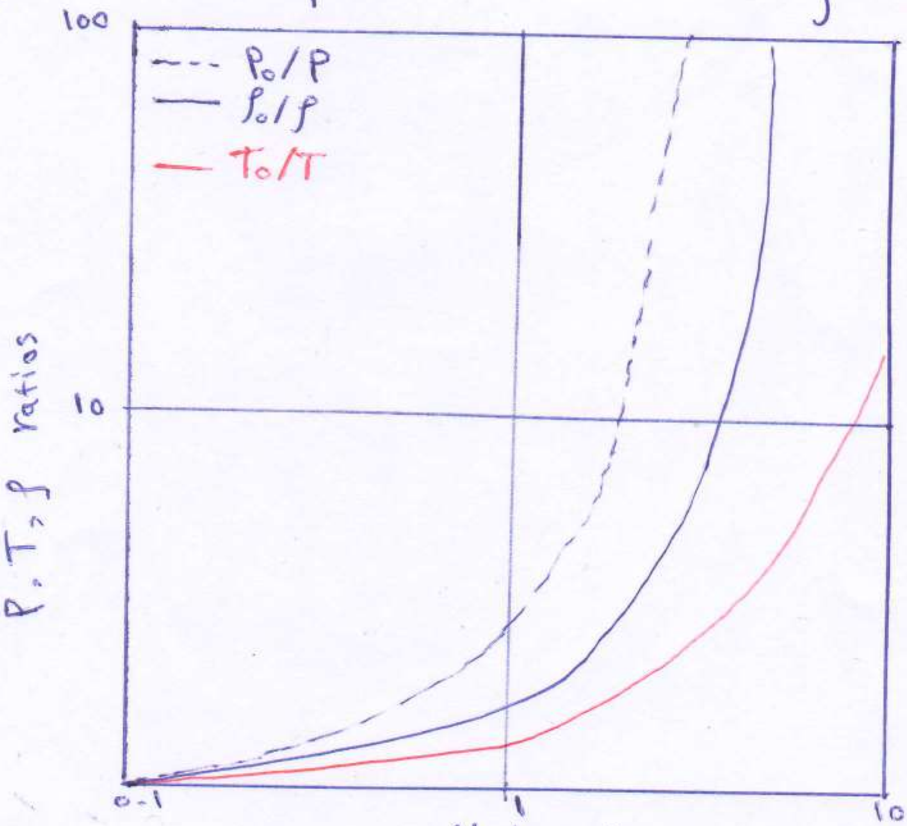


Fig. (2.2)

$S_o = S$ (Isentropic flow)

$$\frac{T_o}{T} = \left[1 + \frac{\gamma-1}{2} M^2\right] \quad , \quad \frac{P_o}{P} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}} \quad , \quad \frac{\rho_o}{\rho} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{1}{\gamma-1}}$$



Mach number

Figure (2.3)

Pressure, Temperature and density ratio Versus Mach number

2-4 Critical flow conditions:

Critical flow conditions are achieved at sonic speed conditions, i.e. when Mach number equals to (1.0). Temperature, pressure and density ratios are given as follows:

$$\frac{T_o}{T^*} = \frac{\gamma+1}{2} \quad , \quad \frac{P_o}{P^*} = \left[\frac{\gamma+1}{2}\right]^{\frac{\gamma}{\gamma-1}} , \quad \frac{\rho_o}{\rho^*} = \left[\frac{\gamma+1}{2}\right]^{\frac{1}{\gamma-1}} , \quad \frac{C_o}{C^*} = \sqrt{\frac{\gamma+1}{2}}$$

2-5 Mass Flow Rate $\dot{m}$:

Substituting from above isentropic flow relations into the continuity equation, the mass flow rate is given as:

$$\dot{m} = \rho A u$$

$$G = \frac{\dot{m}}{A} = \rho u = \frac{P}{RT} \cdot u = \frac{P}{P_o} P_o \cdot \frac{\sqrt{\gamma}}{\sqrt{\gamma R T}} \cdot \frac{u}{\sqrt{RT}}$$

$$= \frac{P}{P_o} \cdot \sqrt{\frac{T_o}{T}} \cdot \frac{P_o}{\sqrt{T_o}} \cdot \sqrt{\frac{\gamma}{R}} \cdot \frac{u}{\sqrt{\gamma R T}}$$

$$G = \frac{\dot{m}}{A} = \frac{P_o}{\sqrt{T_o}} \cdot \sqrt{\frac{\gamma}{R}} \cdot M \left[1 + \frac{\gamma-1}{2} \cdot M^2\right]^{\frac{-(\gamma+1)}{2(\gamma-1)}}$$

then the dimensionless mass flow rate $\frac{\dot{m}}{A} \cdot \frac{\sqrt{T_o}}{P_o} \cdot \sqrt{\frac{R}{\gamma}}$ is given as:

$$\frac{\dot{m}}{A} \cdot \frac{\sqrt{T_o}}{P_o} \cdot \sqrt{\frac{R}{\gamma}} = \frac{M}{\left[1 + \frac{\gamma-1}{2} \cdot M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

2.6 Flow Area Ratio:

For a flow passage, it is more convenient to represent the area in a dimensionless form. So the area of the flow is referenced to the minimum or the critical area of the flow " A^* " as,

$$\frac{A}{A^*} = \frac{(\dot{m}/A^*)}{(\dot{m}/A)} \Rightarrow \frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\cdot\left(1+\frac{\gamma-1}{2}\cdot M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

The area ratio depends only upon Mach number, and it is always grater than one as shown in figure (2-4). For each value of the area ratio A/A^*, there are two correspondent values for Mach numbers; one for subsonic flow and the other for super sonic flow

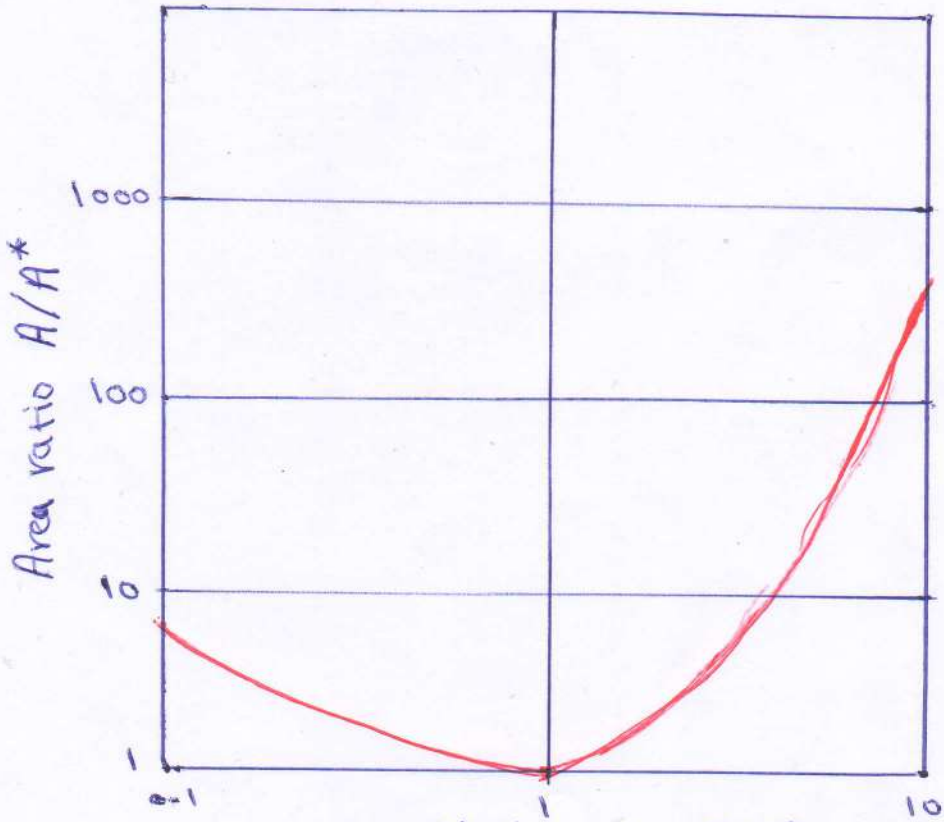


Figure (2-4)
Area ratio versus Mach number

Ex-(2-2)

The gas in a rocket combustion chamber is at a pressure and temperature of 120 Psia and 889 K. The gas expands through an adiabatic frictionless (isentropic) nozzle to a pressure of 15 Psia. Calculate the gas temperature, velocity, Mach number leaving the nozzle. Assume that the gas is a perfect gas and has properties same as that of air.

Solu:

$P_o = 120$ psi $\qquad T_o = 889\ K$

$P_e = 15$ psi $\qquad \gamma = 1.4$

$$\frac{P_o}{P_e} = \frac{120}{15} = 8 = \left[1 + \frac{\gamma-1}{2} M_e^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\Rightarrow M_e = 2.014$$

* ممكن ايجاد M_e من الجدول بمعرفة قيمة $P_e/P_o = 0.125$, مقاطعة الجدول أفقياً لمعرفة M_e

$$\frac{T_o}{T_e} = \frac{889}{T_e} = \left[1 + \frac{\gamma-1}{2} M_e^2\right] \Rightarrow T_e = 490.706\ K$$

* ممكن ايجاد T_e من الجدول بمعرفة قيمة T_e/T_o عند نفس قيمة $M_e = 2.014$

$$C_e = \sqrt{\gamma R T_e} = \sqrt{1.4 * 287 * 490.706}$$

$$\Rightarrow C_e = 444.142\ \frac{m}{s}$$

$$M_e = \frac{U_e}{C_e}$$

$$\Rightarrow U_e = M_e * C_e = 2.014 * 444.142$$

$$U_e = 894.617\ \frac{m}{s}$$

Flow in Variable Area Ducts

Introduction:

Steady one-dimensional compressible flow in varying area ducts or stream tubes are considered in this chapter. Such type of flow is found in many engineering devices such as nozzles, diffusers, flow passages of rocket and airplane engines, and turbo machines. The flow is considered reversible adiabatic (isentropic) except through any shock wave in the passage due to neglecting friction and heat transfer effects.

3.1 Governing Relations for the Flow in a Variable Area Ducts

Consider steady one-dimensional compressible flow in a duct of variable area and of infinitesimal length "dx", as shown in figure (3.1). Infinitesimal variations in flow properties are caused by such area change. Applying the governing equations to the flow through this element, one get;

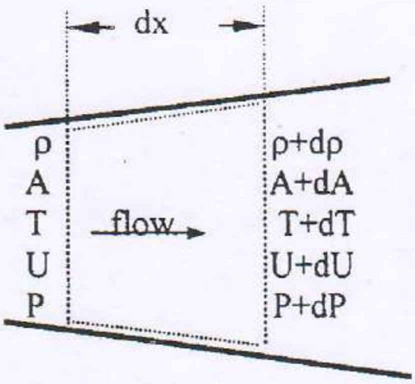


Figure (3.1)
Flow in a variable area duct of infinitesimal length "dx"

3.2 Convergent Divergent Nozzles

Consider the convergent-divergent nozzle shown in figure (3.2a) which is fed from a large reservoir where the stagnation conditions are assumed as P_o, T_o, and ρ_o for pressure, temperature and density respectively. The nozzle discharges into another reservoir where the back pressure is P_B. The back pressure is controlled by means of a control valve. At the exit cross section, flow conditions are denoted by the subscript "E". The flow though the nozzle is assumed to be isentropic. The effect of varying the back pressure on pressure distribution along nozzle axis and flow conditions will be discussed as follows;

Region I:

It includes flow patterns A, B, and C of figure (3.2b), and may be subdivided into two cases (i) and (ii) as follows

i- ***No flow condition (PB=Po):***

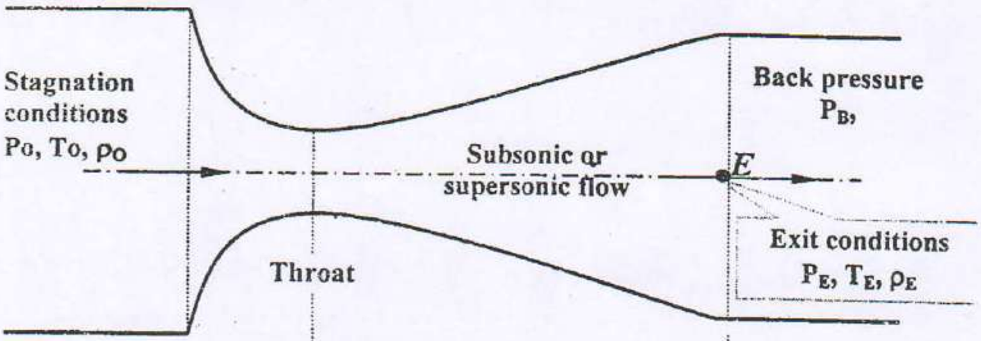
When the back pressure is equal to the upstream (stagnation) pressure P_B=Po, there will be no flow through the nozzle, and the pressure distribution is uniform, as shown in pattern "A" of figure (3.2b), $P_E = P_B = P_o$

ii- ***Subsonic Flow regime (PB $\geq$ PI):***

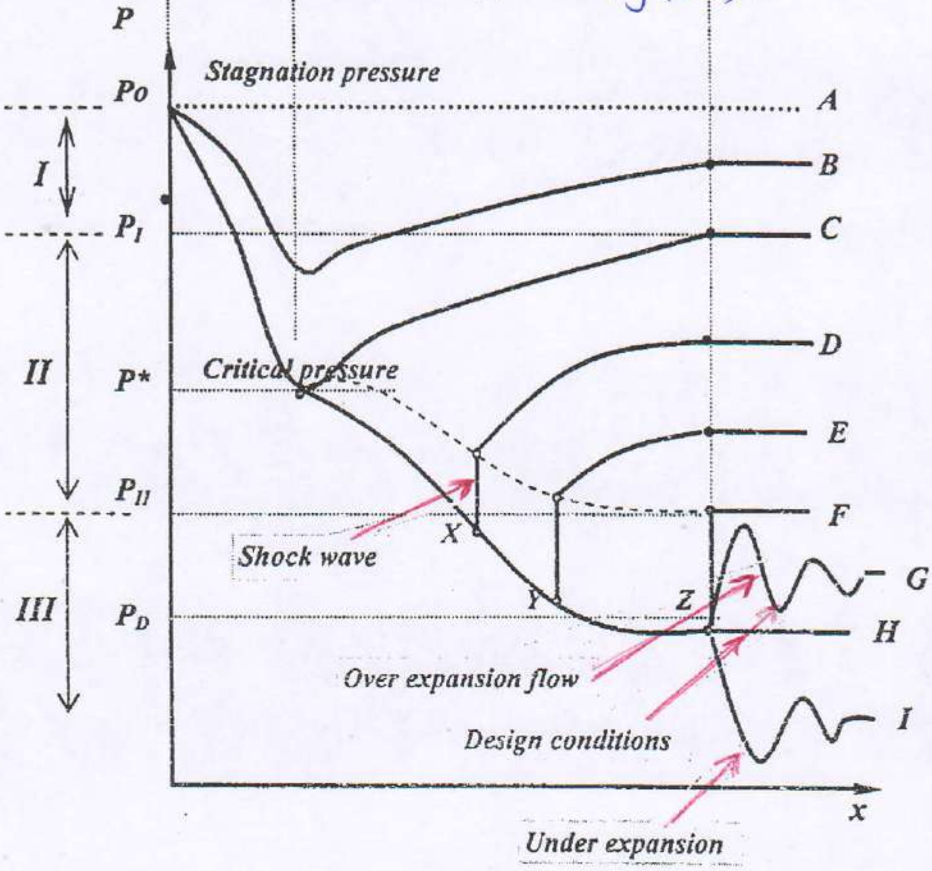
When the back pressure P_B is slightly reduced, then the flow throughout the convergent divergent nozzle is subsonic, as shown in pattern "B" of figure (3.2b). The convergent part of the passage acts as a nozzle where, the pressure decreases and the velocity increases down to the throat section. While the divergent part acts as a diffuser, where, the pressure increases and velocity decreases down to the exit section of the nozzle.

As the back pressure P_B decreases, the mass flow rate and flow velocity throughout the nozzle increase, while the pressure distribution becomes lower. Further reduction in pack pressure will increase the flow velocity and mass flow rate, and lowers the pressure distribution.

At the limit, when P_B=P_I, the flow at the throat becomes sonic, where Mach number is one $M=1.0$, flow velocity is sonic $U=C$, and the pressure reaches the critical value p^*. At this condition the flow rate is maximum, the flow throughout the nozzle is subsonic except at the throat, it is sonic, and the convergent part acts as subsonic nozzle, while the divergent one acts as subsonic diffuser. This is clearly demonstrated by pattern "C" in figure (3.2b).



(a) fig(3.2)a



(b)

Figure (3.2) (3.2)b

Convergent-divergent nozzle performance

Under varying pressure ratio

Region II:

It includes flow patterns D, E and F of figure (3.2b). The flow in this region is supersonic with a shock wave in the divergent part of the nozzle, and it will be discussed in details in the following.

ii- ***Supersonic shocked Flow regime*** ($P_I > P_B \geq P_{II}$):

As the back pressure is reduced below the limiting pressure P_I, the flow pattern in the convergent part of the nozzle is identical to choked flow pattern given by case "C" of figure (3.2b). The flow is accelerating along the convergent part of the nozzle until it reaches sonic conditions at the throat (M=1.0, U=C, P=P*, T=T* and ρ=ρ*). The flow continues to accelerate developing a supersonic flow (M>1.0) downstream the throat in the divergent part of the nozzle. At some section (*X*) downstream in the divergent part, the supersonic flow conditions are terminated by a discontinuity known as shock wave, which changes the flow from supersonic to subsonic. Downstream of the discontinuity, the divergent pat acts as a subsonic diffuser, where the flow velocity decreases and the pressure increases until it ends with a value identical to back pressure at exit cross section. This clearly demonstrated in pattern "D" of figure (3.2b).

Further reduction in back pressure will give the same flow pattern as discussed in the previous paragraph except that the shock wave occurs at cross section (Y) more downstream in the divergent part of the nozzle. This pattern is represented in line "E" of figure (3.2b). For this flow pattern, the flow velocity at exit cross section is subsonic (M<1.0), and flow pressure at exit is identical to back pressure.

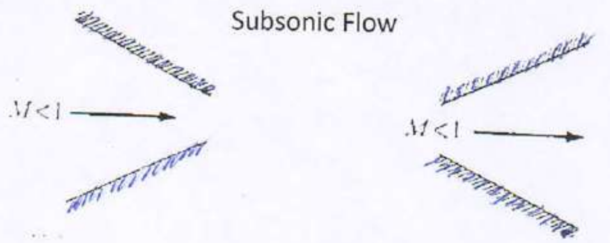
More reduction in back pressure until we reach *the second limiting pressure "P_{II}"* results the same flow pattern as discussed in the previous paragraph except that the shock wave occurs at exit cross section (Z), and the divergent part of the nozzle is a shock free passage. The flow in the divergent part of the nozzle is totally supersonic, however, flow velocity at exit is subsonic due to the shock wave that stands at exit section. Exit pressure is the same as the back pressure. This flow pattern is shown in figure (3.2b) as pattern "F".

3.3 Subsonic and Supersonic Isentropic Flow Through a Varying Area Channel

Nozzle

Diffuser

Subsonic Flow



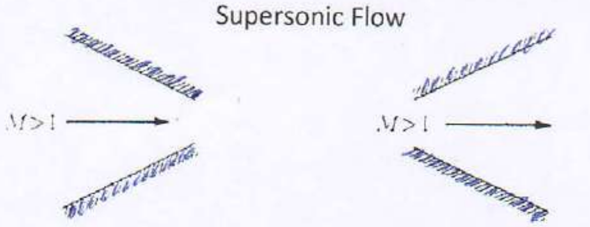
Velocity increases
Pressure decreases

Velocity decreases
Pressure increases

Nozzle

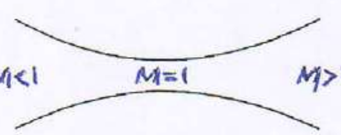
Diffuser

Supersonic Flow



Velocity decreases
Pressure increases

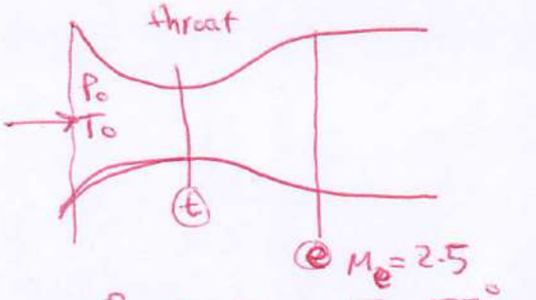
Velocity increases
Pressure decreases



ex (3.1)

A supersonic wind tunnel nozzle is to be designed for $M = 2.5$, with a throat section of $5\ cm^2$. The supply pressure and temperature at inlet, where the velocity is nigligible are 200 KPa and 137 °C respectively. Assuming that the flow is one-dimensional isentropic, find the mass flow rate, test section area, and fluid properties at throat and test section.

Solu.



* at $M = 1$ at throat section

then $\dfrac{\dot m}{A_t}\dfrac{\sqrt{T_o}}{P_o}\sqrt{\dfrac{R}{\gamma}} = 0.5787$

$$\Rightarrow \frac{\dot m}{5*10^{-4}} \cdot \frac{\sqrt{410}}{200*10^3}\sqrt{\frac{287}{1.4}} = 0.5787$$

$$\Rightarrow \boxed{\dot m = 0.2\ kg/s}$$

$$\rho_o = \frac{P_o}{RT_o} = \frac{200*10^3}{287*410} \Rightarrow \rho_o = 1.7\ kg/m^3$$

$$\frac{P_t}{P_o} = 0.52828 \Rightarrow \boxed{P_t = 105.66\ kpa}$$

$$\rho_t/\rho_o = 0.63394 \Rightarrow \boxed{\rho_t = 1.0777\ kg/m^3}$$

$$T_t/T_o = 0.83333 \Rightarrow \boxed{T_t = 341.67\ K}$$

* at $M_e = 2.5$

$$\Rightarrow A_e/A^* = 2.63672 \Rightarrow \boxed{A_e = 13.184\ cm^2}$$

$$P_e/P_o = 0.05853 \Rightarrow \boxed{P_e = 11.71\ KPa}$$

$$\rho_e/\rho_o = 0.13169 \Rightarrow \boxed{\rho_e = 0.2239\ kg/m^3}$$

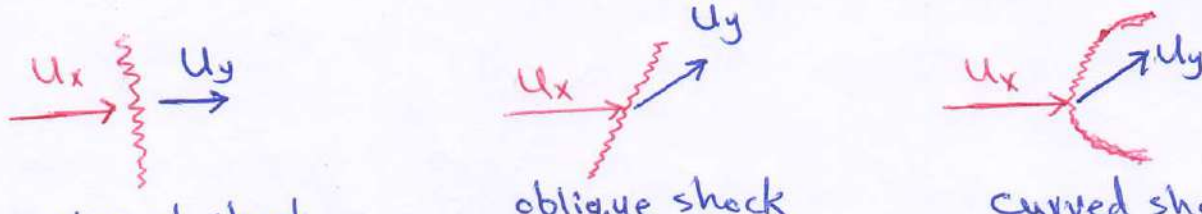
$$T_e/T_o = 0.44444 \Rightarrow \boxed{T_e = 182.22\ K}$$

$$C_e = \sqrt{\gamma R T_e} = \sqrt{1.4*287*182.22} \Rightarrow C_e = 270.58\ m/s$$

$$V_e = M_e * C_e = 2.5 * 270.58 \Rightarrow \boxed{V_e = 676.45\ m/s}$$

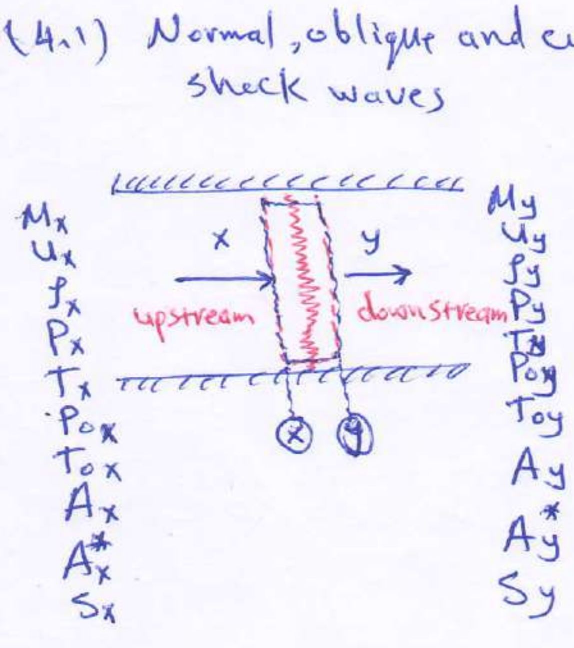
Normal Shock Waves

Normal shock wave is a phenomenon in gas supersonic flows, where a sudden change of fluid properties occurs across a very small region (of order 10^{-5} cm thickness), the pressure increases across the shock wave and the velocity decreases.



Normal shock wave — oblique shock wave — curved shock wave

fig. (4.1) Normal, oblique and curved shock waves



$M_x > 1$ (شرط الصدمة)

$M_x > M_y$, $U_x > U_y$

$\rho_x < \rho_y$

$T_x < T_y$

$T_{ox} = T_{oy}$

$P_{ox} > P_{oy}$

$P_y > P_x$

$A_x = A_y$ (لكون منطقة الصدمة صغيرة جداً)

$A_x^* \neq A_y^*$, $A_x^* < A_y^*$

A_x^* : تمثل المساحة الحرجة التي يحصل عندها (M=1) في منطقة بين الدخول (section 1) ومنطقة قبل الصدمة (section x) $(1 \rightarrow x)$

A_y^* : تمثل المساحة الحرجة التي يحصل عندها (M=1) في منطقة ما بين بعد الصدمة (section y) ومنطقة الخروج (section 2) $(y \rightarrow 2)$

$S_y > S_x$

Governing Equations of Normal shock wave

Energy Equation:

$$h_x + \frac{U_x^2}{2} = h_y + \frac{U_y^2}{2} = h_o$$

Continuity Equation:

$$\rho_x U_x = \rho_y U_y = \dot{m}/A \qquad \dot{m} = \dot{m}_x = \dot{m}_y$$

Momentum Equation:

$$P_x - P_y = \frac{\dot{m}}{A}(U_x - U_y)$$

or

$$P_x + \rho_x U_x^2 = P_y + \rho_y U_y^2$$

State Equation:

$$P_x = \rho_x R T_x \Rightarrow R = \frac{P_x}{\rho_x T_x}$$

$$P_y = \rho_y R T_y \Rightarrow R = \frac{P_y}{\rho_y T_y}$$

$$\Rightarrow \frac{P_x}{\rho_x T_x} = \frac{P_y}{\rho_y T_y} \quad , \quad \frac{\rho_y}{\rho_x} = \frac{P_y}{P_x} \cdot \frac{T_x}{T_y}$$

* Mach number change across shock wave

$$\boxed{M_y^2 = \frac{M_x^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma}{\gamma - 1} \cdot M_x^2 - 1}}$$

* Pressure Ratio Across Normal Shock Wave

$$\boxed{\frac{P_y}{P_x} = \frac{2\cdot\gamma}{\gamma+1}\cdot M_x^2 - \frac{\gamma-1}{\gamma+1}}$$

* Temperature Ratio Across Normal Shock wave

$$\boxed{\frac{T_y}{T_x} = \frac{\left(1+\frac{\gamma-1}{2}\cdot M_x^2\right)\left(\frac{2\gamma}{\gamma-1}\cdot M_x^2-1\right)}{\frac{(\gamma+1)^2}{2(\gamma-1)}\cdot M_x^2}}$$

* Density and Velocity Ratios Across Normal Shock wave

$$\boxed{\frac{U_y}{U_x} = \frac{\rho_x}{\rho_y} = \frac{2+(\gamma-1)M_x^2}{(\gamma+1)M_x^2}}$$

* Critical Area Ratio Across Normal Shock wave

$$\boxed{\frac{A_x^*}{A_y^*} = \frac{P_{oy}}{P_{ox}} = \left[\frac{\frac{\gamma+1}{2}M_x^2}{1+\frac{\gamma-1}{2}M_x^2}\right]^{\frac{\gamma}{\gamma-1}}\cdot\left(\frac{2\gamma}{\gamma+1}M_x^2-\frac{\gamma-1}{\gamma+1}\right)^{\frac{-1}{\gamma-1}}}$$

* Entropy change Across Normal Shock wave $(S_y \neq S_x)$

$(S_1 = S_x)$ & $(S_y = S_2)$

$$\boxed{\frac{S_y-S_x}{R} = \frac{\gamma}{\gamma-1}\ln\left(\frac{2}{(\gamma+1)M_x^2}+\frac{\gamma-1}{\gamma+1}\right)+\frac{1}{\gamma-1}\ln\left(\frac{2\gamma}{\gamma+1}M_x^2-\frac{\gamma-1}{\gamma+1}\right)}$$

Strength of a Shock Wave :

The strength of a normal shock wave Φ is defined as the ratio of the pressure rise across the shock wave to the initial pressure befor shock

$$\Phi = \frac{P_y - P_x}{P_x} = \frac{P_y}{P_x} - 1$$

$$\boxed{\Phi = \frac{2\gamma}{\gamma+1} \cdot \left(M_x^2 - 1\right)}$$

Ex (4.1)

A normal shock occurs at a point in airflow where the Pressure and temperature are 50 kPa , 263 K respectively. If the Pressure rise across the shock is 2.7 , find :

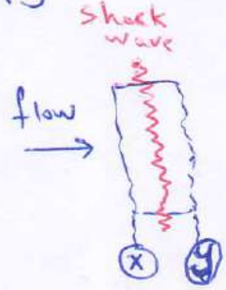
1- The pressure and temperature downstream the shock wave.
2- Mach numbers and flow velocities upstream and downstream the shock wave.
3- change in stagnation pressure and entropy across shock wave.

Solu: $P_x = 50\ kPa$, $T_x = 263\ K$

$P_y/P_x = 2.7$ $\quad P_y = ?$ $\quad T_y = ?$

$M_x = ?$ $\quad u_x = ?$ $\quad M_y = ?$ $\quad u_y = ?$

$\Delta P_o = ?$ $\quad \Delta S = ?$



from table C & $P_y/P_x = 2.7$

$$\Rightarrow \boxed{M_x = 1.57} \ , \ \boxed{M_y = 0.67769} \ , \ \frac{u_x}{u_y} = 1.98119$$

$$\frac{T_y}{T_x} = 1.36738 \ , \ \frac{P_{oy}}{P_{ox}} = 0.90615 \ , \ \frac{S_y - S_x}{R} = 0.09855$$

$$P_y = 2.7 P_x \Rightarrow \boxed{P_y = 135\ kPa}$$

$$T_y = 1.36738\, T_x \Rightarrow \boxed{T_y = 359.62\ k}$$

$U_x = M_x \cdot \sqrt{\gamma R T_x} \Rightarrow \boxed{U_x = 510.37 \text{ m/s}}$

$U_y = U_x / 1.98119 \Rightarrow \boxed{U_y = 257.61 \text{ m/s}}$

* from table B at $M_x = 1.57$

$\Rightarrow P_x / P_{o_x} = 0.24593 \Rightarrow \underline{P_{ox} = 203.31 \text{ KPa}}$

$P_{oy} = 0.90615\ P_{ox} \Rightarrow \underline{P_{oy} = 184.23 \text{ KPa}}$

$\Delta P_o = P_{ox} - P_{oy} \Rightarrow \boxed{\Delta P_o = 19.1 \text{ KPa}}$

$\Delta S = S_y - S_x = 0.09855 * R \Rightarrow \boxed{\Delta S = 28.28 \text{ J/kg}\cdot\text{K}}$

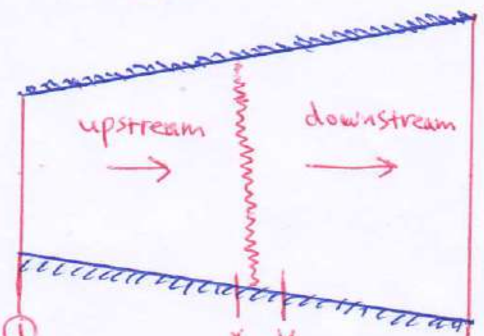
Ex(4-2)

An airstream at Mach 2.0 with pressure of 100 KPa and temperature of 270 K, enters a divergent channel, with a ratio of exit area to inlet area of 3.0. Determine the back pressure necessary to produce a normal shock in the channel at an area equal to twice the inlet area.

Solu:

$M_1 = 2 \quad P_1 = 100 \text{ KPa} \quad T_1 = 270 \text{ K}$

$\frac{A_2}{A_1} = 3 \ , \ \frac{A_x}{A_1} = 2 \qquad P_2 = P_b = ?$



* from table B & $M_1 = 2$

$\frac{A_1}{A_1^*} = 1.6875 \ , \ \frac{P_1}{P_{o_1}} = 0.1278$

$(A_y^* = A_2^*)$ & $(A_1^* = A_x^*)$

* لمعرفة M_x يجب معرفة النسبة $\frac{A_x}{A_x^*}$ & جدول الجريان الايزنتروبي (B)

نختار $M_x > 1$ وليس $\cancel{M < 1}$

$$\frac{A_x}{A_x^*} = \frac{A_x}{A_1} \cdot \frac{A_1}{A_1^*} = 2 * 1.6875 = 3.375$$

* from table B & $A_x / A_x^* = 3.375$

$\Rightarrow M_x = 2.76$

$\Rightarrow P_{o_1} = 100 / 0.1278 = 782.5 \text{ KPa}$

* from table B & $M_x = 2.76$

$P_x / P_{ox} = 0.3917$, $(P_{ox} = P_{o1})$

$\Rightarrow P_x = 30.65$ kPa

* from table C & $M_x = 2.76$

$\Rightarrow M_y = 0.49107$, $\frac{P_y}{P_x} = 8.72053$, $\frac{P_{oy}}{P_{ox}} = 0.40283$

$\Rightarrow P_y = 267.28$ kPa , $P_{oy} = 315.21$ kPa $= P_{o2}$

* لمعرفة قيمة M_2 يجب معرفة النسبة $\frac{A_2}{A_2^*}$ ومقاطعتها مع جدول الجريان الايزنتروبي B حيث يتم اختيار $(M_2 < 1)$ وليس $(M_2 > 1)$ وذلك لان المقطع السابق (y) فيه $(M_y < 1)$ وان الشكل الهندسي لقناة الجريان تباعدية بمعنى ان قيمة M ستقل مع تقدم الجريان من المقطع y الى المقطع 2

* from table B & $M_y \cong 0.49$

$\Rightarrow \frac{A_y}{A_y^*} = 1.35947$ $\qquad (A_y^* = A_2^*)$

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1} * \frac{A_y}{A_y^*} * \frac{A_1}{A_x} = 3 * 1.35947 * \frac{1}{2} \quad (A_x = A_y)$$

$\Rightarrow \frac{A_2}{A_2^*} = 2.0392$

* from table B & $\frac{A_2}{A_2^*}$, find M_2 $\qquad (M_2 < 1)$

$\Rightarrow M_2 = 0.3$, $\frac{P_2}{P_{o2}} = 0.93947$

~~or $M_2 = 2.22$ مرفوض للسبب اعلاه~~

for subsonic Mach number $M_2 = 0.3$

$\Rightarrow \boxed{P_2 = P_{back} = 296.13 \text{ kPa}}$

Moving Normal Shock wave

In the foregoing sections, Stationary normal shock waves are discussed, However, in many practical situations normal shock waves are moving relative to a fixed reference coordinates. Moving shock waves are encountered in explosions, gas pipes, shock tubes and inlet and exhaust system of internal combustion engines.

If the shock where travels at constant speed (U_s), the analysis of the problem may be reduced to a stationary normal shock wave by employing a moving coordinate system.

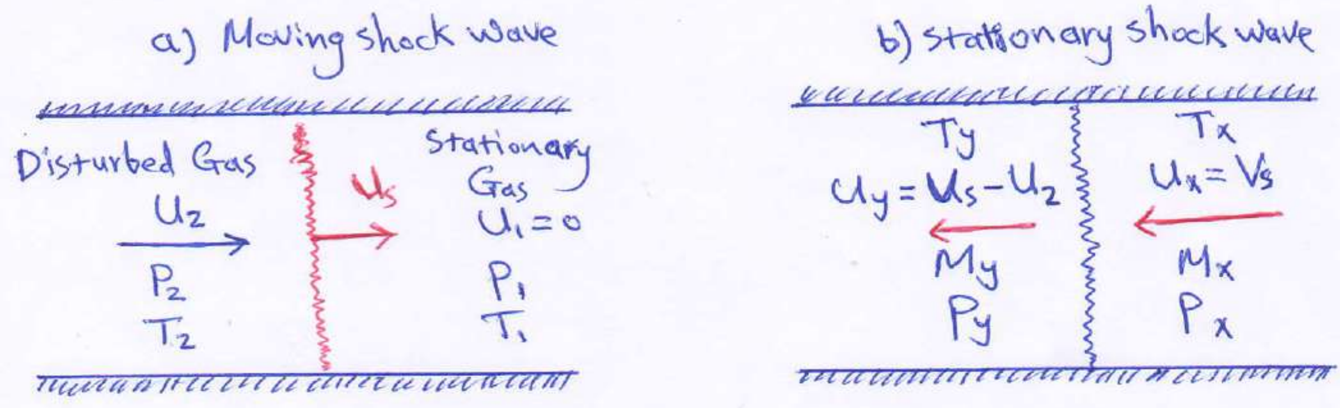


Figure (4.2)
Transformation of moving normal shock wave into stationary normal shock wave

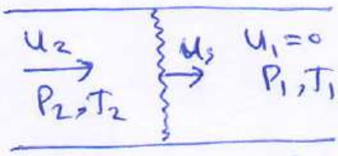
Ex (4.3)

A shock wave across which the pressure ratio is 1.45 is moving into still air at a pressure of 100 kPa, and temperature of 298 K, Find the velocity, pressure and temperature of air behind the shock wave.

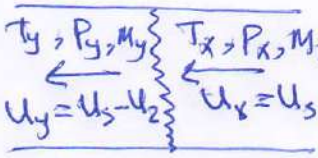
Solu: $\frac{P_y}{P_x} = 1.45$

$P_1 = 100\,kPa$, $T_1 = 298\,K$

$U_2 = ?$, $P_y = ?$, $T_y = ?$



a) moving shock



b) stationary shock

* from table c & $P_y/P_x = 1.45$

$$\Rightarrow M_x = 1.175 \;,\; \frac{U_x}{U_y} = 1.29827 \;,\; \frac{T_y}{T_x} = 1.1123$$

$$U_x = M_x * \sqrt{\gamma R T_x} \Rightarrow U_x = 406.58\ m/s$$

$$U_y = \frac{U_x}{1.29827} \Rightarrow U_y = 313.17\ m/s$$

$$U_y = U_s - U_2 \;,\; U_s = U_x$$

$$\Rightarrow U_2 = U_x - U_y \Rightarrow \boxed{U_2 = 93.41\ m/s}$$

$$P_y = 1.45\,P_x \Rightarrow \boxed{P_y = 145\ kPa = P_2}$$

$$T_y = 1.1123\,T_x \Rightarrow \boxed{T_y = 331.47\,K = T_2}$$

Ex.(4.4) الموجة العمودية (متحركة)

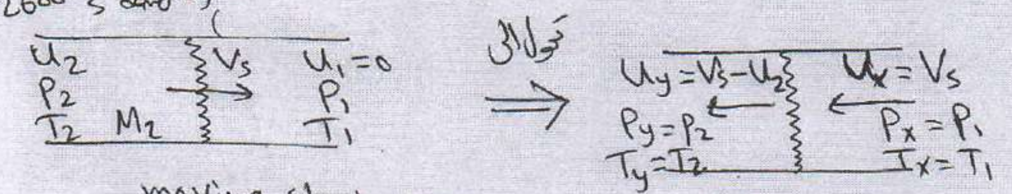
A normal shock wave propagates at a speed of 2600 $\frac{m}{s}$ down a pipe that is filled with hydrogen. The hydrogen is at rest and at a pressure and temperature of 101.3 Kpa and 25 °C respectively upstream of the wave. Assuming hydrogen to behave as a perfect gas with constant specific heats, find the temperature, pressure and velocity downstream of the shock wave.

take $\gamma = 1.405$ & $R = 4124$ J/Kg.K

المطلوب find (T_y, P_y, U_2 =?)

① يتم تحويل الموجة المتحركة الى موجة ثابتة

$V_s = 2600\ \frac{m}{s}$ سرعة الموجة



moving shock wave — Stationary shockwave

الضغط ودرجة الحرارة في حالة الراحة at rest اي قبل ان تتحرك الموجة يعني ظروف الحالة x قبل الموجة

$$P_x = P_1 = 101.3\ kpa \qquad T_x = T_1 = 25 + 273 = 298\ K$$

$$M_x = \frac{U_x}{C_x} = \frac{U_x}{\sqrt{\gamma R T_x}} = \frac{2600}{\sqrt{1.405 * 4124 * 298}} , \boxed{M_x = 1.978}$$

باستخدام القوانين ص 42

$$M_y^2 = \frac{M_x^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1} M_x^2 - 1} \Rightarrow \boxed{M_y = 0.5817}$$

$$\frac{P_y}{P_x} = \frac{2\gamma}{\gamma+1} M_x^2 - \frac{\gamma-1}{\gamma+1} = 4.4059 , \boxed{P_y = 446.32\ kpa}$$

من الجداول $\frac{T_y}{T_x} = 1.6778 \Rightarrow \boxed{T_y = 500\ K}$

$$C_y = \sqrt{\gamma R T_y} , U_y = C_y M_y \Rightarrow U_y = 990.1\ \frac{m}{s}$$

* ارجاع الحالة الى الاولى تحويل الموجة الثابتة الى متحركة *

$$\Rightarrow P_2 = P_y = 446.32\ KPa$$
$$T_2 = T_y = 500\ K$$
$$U_2 = V_s - U_y = 2600 - 990.1 \Rightarrow \boxed{U_2 = 1609.9\ \frac{m}{s}}$$

Mohammed Gh. Jehad

Oblique Shock Wave

oblique shock waves are essentially straight waves but they are at an angle to the upstream flow. oblique shock wave occur: when a wedge-shape object is placed in a two-dimensional supersonic flow. They may be attached or detached to the object nose. The former is an oblique shock while later is a curved one stands in front of the object, as shown in figure(5.1).

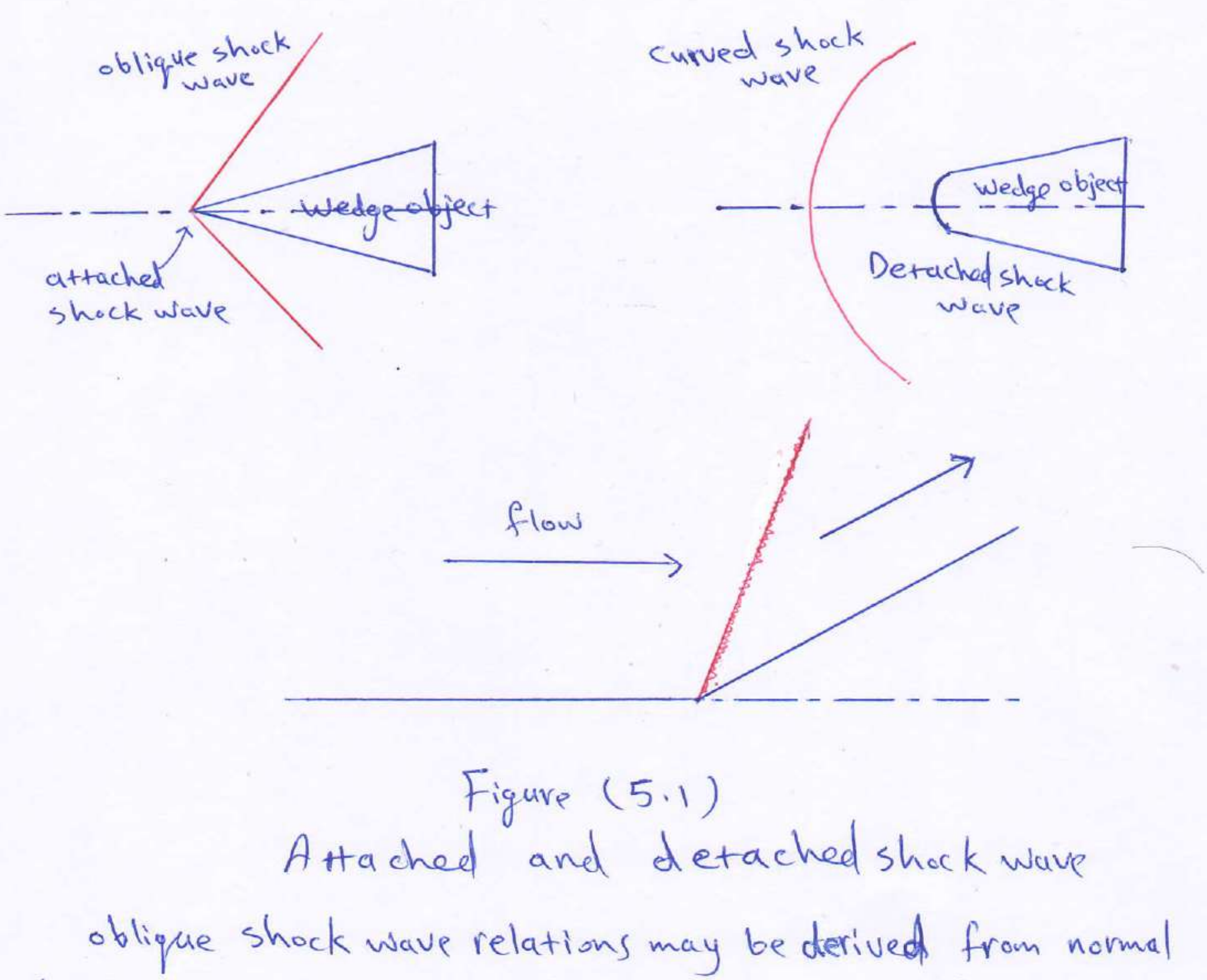
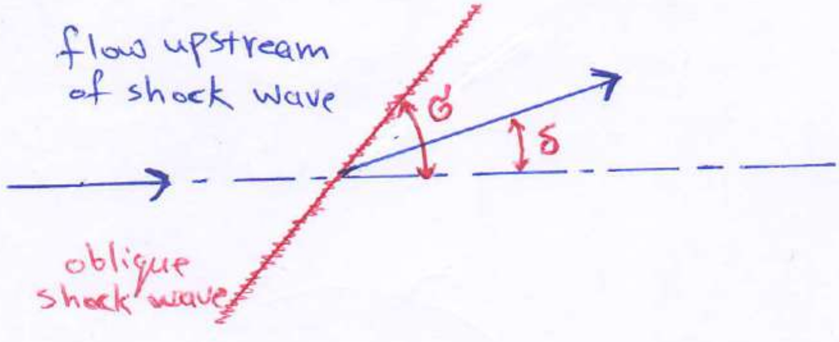


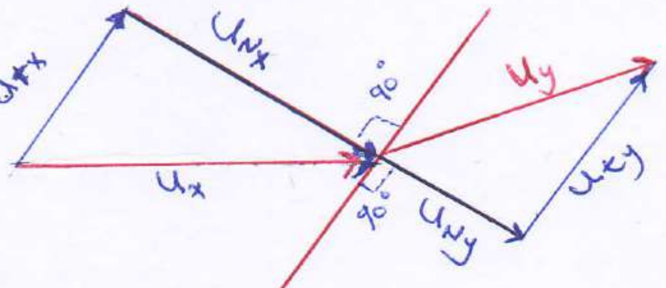
Figure (5.1)
Attached and detached shock wave

oblique shock wave relations may be derived from normal shockwave relations. As there are no changes in flow varibles in a direction parallel to the shock.

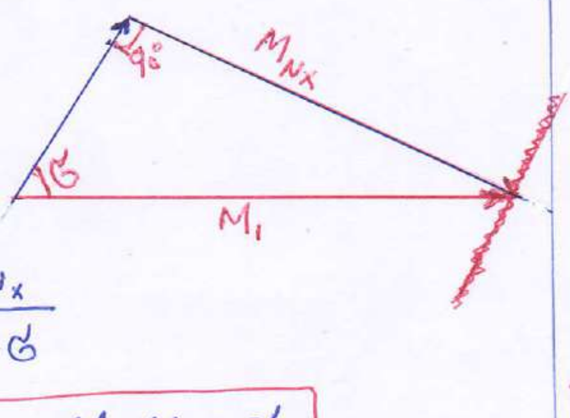
oblique Shock Wave Governing Relations

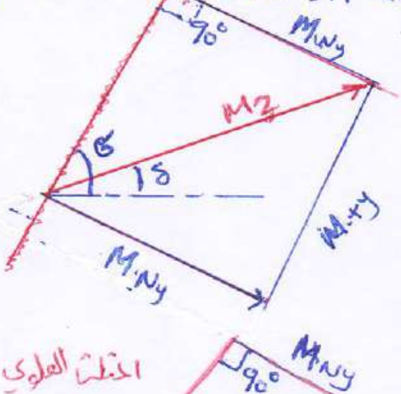
as shown in figure (5.2) the flow is deflected an angle "δ" after it passes the shock. The shock itself makes an angle "σ" with the upstream flow velocity.





Figure(5.2) velocity components upstream and downstream the shock





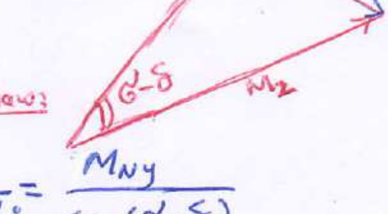
Sin law:

$$\frac{M_1}{\sin 90} = \frac{M_{N_x}}{\sin \sigma}$$

$$\Rightarrow \boxed{M_{N_x} = M_1 \sin \sigma}$$

σ: shock wave angle

δ: flow deflection angle



Sine law:

$$\frac{M_2}{\sin 90°} = \frac{M_{Ny}}{\sin(\sigma-\delta)}$$

$$\Rightarrow \boxed{M_2 = \frac{M_{Ny}}{\sin(\sigma-\delta)}}$$

$$\boxed{\tan(\delta) = \frac{2\left(M_x^2 \sin^2(\sigma) - 1\right)}{2 + M_x^2 \cdot \left(\gamma + \cos(2\sigma)\right)} \cdot \cot(\sigma)}$$

$$\boxed{M_y^2 \cdot \sin^2(\sigma - \delta) = \frac{M_x^2 \cdot \sin^2(\sigma) + \frac{2}{\gamma - 1}}{\frac{2\gamma}{\gamma - 1} \cdot M_x^2 \cdot \sin^2(\sigma) - 1}}$$

$$\boxed{\frac{P_y}{P_x} = \frac{2\gamma}{\gamma + 1} \cdot M_x^2 \cdot \sin^2(\sigma) - \frac{\gamma - 1}{\gamma + 1}}$$

$$\boxed{\frac{T_y}{T_x} = \frac{\left(1 + \frac{(\gamma - 1)}{2} \cdot M_x^2 \sin^2(\sigma)\right) \cdot \left(\frac{2\gamma}{\gamma - 1} \cdot M_x^2 \cdot \sin^2(\sigma) - 1\right)}{\frac{(\gamma + 1)^2}{2(\gamma - 1)} \cdot M_x^2 \cdot \sin^2(\sigma)}}$$

$$\boxed{\frac{\rho_y}{\rho_x} = \frac{(\gamma + 1) \cdot M_x^2 \cdot \sin^2(\sigma)}{2 + (\gamma - 1) \cdot M_x^2 \cdot \sin^2(\sigma)}}$$

$$\boxed{\frac{P_{oy}}{P_{ox}} = \left(\frac{\frac{\gamma + 1}{2} M_x^2 \sin^2(\sigma)}{1 + \frac{\gamma - 1}{2} M_x^2 \sin^2(\sigma)}\right)^{\frac{\gamma}{\gamma - 1}} \cdot \left(\frac{P_y}{P_x}\right)^{\frac{-1}{\gamma - 1}}}$$

$$\boxed{\begin{aligned} \frac{S_y - S_x}{R} &= \frac{\gamma}{\gamma - 1} \ln\left(\frac{2}{(\gamma + 1) M_x^2 \cdot \sin^2(\sigma)} + \frac{\gamma - 1}{\gamma + 1}\right) \\ &\quad + \frac{1}{\gamma - 1} \cdot \ln\left(\frac{2\gamma}{\gamma + 1} \cdot M_x^2 \sin^2(\sigma) - \frac{\gamma - 1}{\gamma + 1}\right) \end{aligned}}$$

The minimum oblique shock wave angle (σ_{min})

$$\boxed{\sigma_{min} = \sin^{-1}(1/M_x)}$$

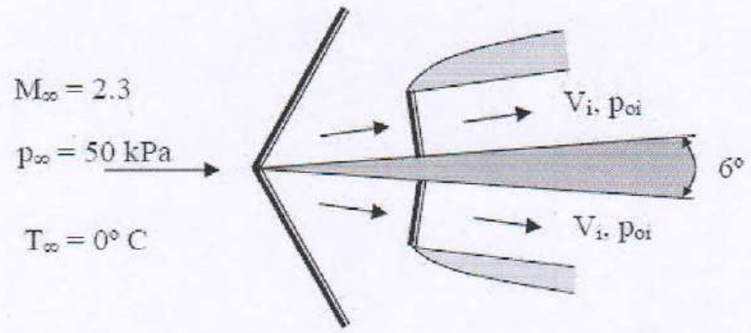
The maximum shock wave angle (σ_{max}) corresponds to themaximum deflection Angle (δ_{max})

$$\boxed{\sin^2(\sigma_{max}) = \frac{1}{\gamma M_x^2}\left(\frac{\gamma+1}{4} M_x^2 - 1 + \sqrt{Z}\right)}$$

where Z is :

$$Z = (\gamma+1)\left(1 + \frac{\gamma-1}{2} M_x^2 + \frac{\gamma+1}{16} M_x^4\right)$$

H.W.1 – For the two-dimensional diffuser shown in Figure, find V_i and p_{oi}.



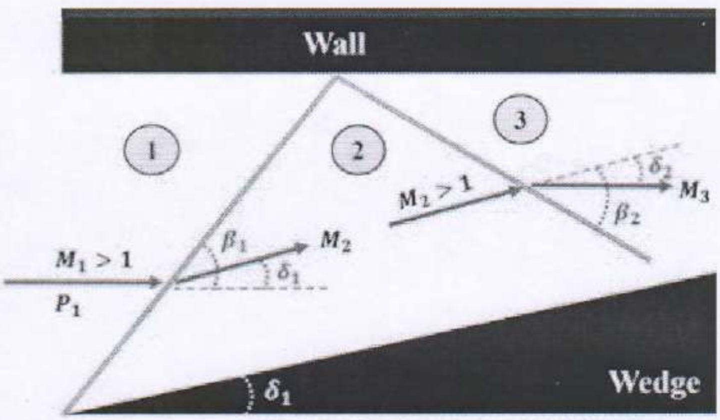
H.W.2

Knowns:

- $M_1 = 3$
- $P_1 = 100$ kPa
- $\delta_1 = \delta_2 = 15°$

Unknowns:

- $P_2 = ?$
- $P_3 = ?$



EX (5-1)

((الصدمة المائلة))

A supersonic air flows at Mach number of 3.2 Passes over a concave corner. An oblique shock wave, which makes an angle of 35° with flow direction, is attached to the corner under the given conditions. The pressure and temperature of the flow are 21.5 Kpa and 232 K respectively.

1- Determin the Pressure and temperature behind shock wave.

2- Determine also the downstream Mach number and deflection angle.

solu:

$M_x = 3.2 \qquad \sigma = 35^\circ$

requ: P_y, T_y, M_y, δ

$\gamma = 1.4$, $R = 287\ J/kg.K$

$P_x = 21.5\ kpa$

$T_x = 232\ k$

$\frac{T_{ox}}{T_x} = 1 + \frac{\gamma-1}{2} M_x^2 \Rightarrow T_{ox} = 707.136\ K$ من جداول الجريان الايزنتروبي وعند $M_x = 3.2$

$\frac{P_{ox}}{P_x} = \left(1 + \frac{\gamma-1}{2} M_x^2\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow P_{ox} = 1063\ kpa$

$C_x = \sqrt{\gamma R T_x} \Rightarrow C_x = 305.3\ \frac{m}{s}$, $U_x = C_x M_x \Rightarrow U_x = 977\ \frac{m}{s}$

* الصدمة المائلة نجد رقم ماخ العمودي في الصدمة M_{xn}

قانون يحفظ $\boxed{M_{xn} = M_x \sin \sigma} \Rightarrow M_{xn} = 1.835$

* من جداول الصدمة العمودية وعند $M_{xn} = 1.835$

$\frac{P_y}{P_x} = 3.764 \Rightarrow \boxed{P_y = 80.926\ kpa}$ $\qquad \frac{80.926}{101.325} = 0.799\ atm$ تحويل وحدات الضغط (اختياري)

$\frac{T_y}{T_x} = 1.558 \Rightarrow \boxed{T_y = 361.5\ k}$, $M_{yn} = 0.609$

يمكن استخراج زاوية الانحراف δ من الجداول او مخططات الصدمة المائلة بدلالة $(M_x = 3.2)$ & σ زاوية ميل الصدمة مع خط جريان المائع. او من الرسم صفحة 91 شكل (D-1)

$\Rightarrow \delta = 18.833^\circ$

نرجع الى الجريان الحقيقي (المائل)

$M_y = \frac{M_{yn}}{\sin(\sigma - \delta)} = \frac{0.609}{\sin(35 - 18.833)}$

$\boxed{M_y = 2.187}$

Mohammed Gh. Jehad

EX (5-2)   ((الصدمة المائلة))

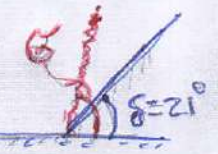
A supersonic air flow of Mach number 2.8 passes over a concave corner which deflects the flow by 21°. The pressure and temperature of the flow are 0.65 atm and 257 K respectively. Determine the flow Mach number, pressure and temperature downstream of the oblique shock, and the maximum deflection angle.

Find M_y, P_y, T_y; δ_{max}

Sol.

$M_x = 2.8 \quad P_x = 0.65\ atm$

$\xrightarrow{flow} T_x = 257\ K$



* من الجدول او العلاقات او الرسم للصدمة المائلة ومن معرفة

$\sigma = 40.72^\circ \Leftarrow M_x = 2.8 \ \& \ \delta\ 21^\circ$

$M_{xn} = M_x \sin\sigma \Rightarrow \boxed{M_{xn} = 1.827}$

* من جدول الجريان الايزنتروبي وعند $M_x = 2.8$

$$\frac{T_{ox}}{T_x} = 1 + \frac{\gamma-1}{2} M_x^2 \Rightarrow T_{ox} = 660\ K$$

$$\frac{P_{ox}}{P_x} = \left(1 + \frac{\gamma-1}{2} M_x^2\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow P_{ox} = 17.64\ atm$$

$$C_x = \sqrt{\gamma R T_x} \Rightarrow C_x = 321.35\ \frac{m}{s}, \quad U_x = C_x M_x \Rightarrow U_x = 899.8\ \frac{m}{s}$$

* من جداول الصدمة الرأسية وعند $M_{xn} = 1.827$

$$M_{yn} = 0.611, \quad M_y = \frac{M_{yn}}{\sin(\sigma-\delta)} \Rightarrow \boxed{M_y = 1.81}$$

$$\frac{P_y}{P_x} = 3.726 \Rightarrow \boxed{P_y = 2.422\ atm}$$

$$\frac{T_y}{T_x} = 1.552 \Rightarrow \boxed{T_y = 398.75\ K}$$

Mohammed Gh. Jehad

Frictional Adiabatic Flow in Constant Area Duct
(Fanno Flow)

Compressible fluid flow in constant area ducts is important for most engineering fields, such as natural gas transport, ducting systems in power stations. In real fluid flows, wall shear stress and frictional effects are present and have significant influences on flow characteristics. The influence of viscosity is assumed nigligible in nozzles, diffusers, or short ducts. But for long ducts, viscous effect are significant, wher the frictional adiabatic flow in constant area ducts is considered.

Governing Equations:

Consider the steady one-dimensional adiabatic frictional flow through a control volume of length (dx) of the constant area duct shown in figure (6-1)

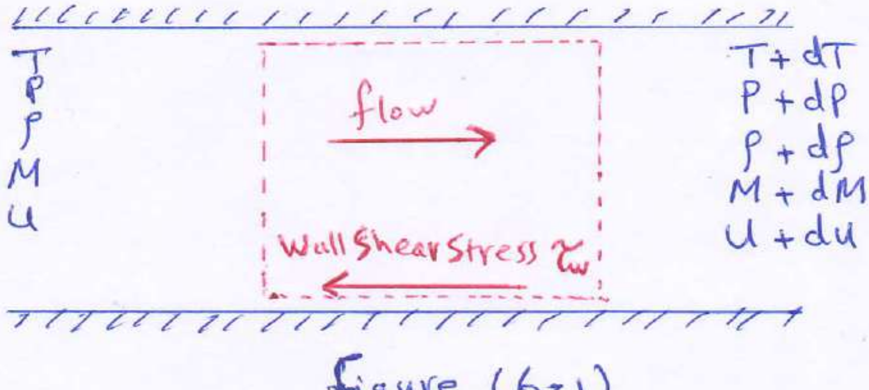


Figure (6-1)
Control Volume for adiabatic frictional flow through constant area duct.

* Continuity Equation :

Since the area is constant, the mas flowrate per unit area is given as:

$$\frac{\dot{m}}{A} = \rho . u = (\rho + d\rho) . (u + du)$$

$$\Rightarrow \quad \frac{d\rho}{\rho} + \frac{du}{u} = 0$$

* Momentum equation :

$$- \tau_w . dA_w + P . A - (P + dP) . A = (\rho + d\rho)(u + du)^2 A - \rho u^2 . A$$

Neglecting the second order diffential terms

$$\Rightarrow - \tau_w \, dA_w - dP . A = \rho u A \; du$$

A = cross sectional area of the duct

τ_w = the wall shear stress

dA_w = the wetted area of the duct which related with D

D = hydraulic diameter , D = 4 cross sectional area/premeter

f = the friction coefficient

$$f = \frac{\tau_w}{\frac{1}{2} \rho u^2}$$

$$\Rightarrow \quad \frac{dP}{P} + \frac{\gamma M^2}{2} . \frac{4 f \, dx}{D} + \gamma M^2 . \frac{du}{u} = 0$$

* Energy Equation : (First law of thermodynamics)

$$\frac{dT}{T} + (\gamma - 1) . M^2 . \frac{du}{u} = 0$$

Equation of State :

$$\frac{P}{\rho R T} = \text{Constant}$$

after differentiation, we get the following relation

$$-\frac{dP}{P} + \frac{dT}{T} + \frac{d\rho}{\rho} = 0$$

Definition of Machnumber

$$M^2 = u^2 / c^2 = u^2 / \gamma R T$$

after differentiation, we get the following

$$2\frac{du}{u} - \frac{dT}{T} = \frac{dM^2}{M^2}$$

Fanno Flow working Formula and Relations

The maximum Frictional length of the duct in Fanno flow $(4f L_{max}/D)$:

$$\boxed{\frac{4f L_{max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma} \ln\left(\frac{\frac{\gamma+1}{2} M^2}{1+\frac{\gamma-1}{2} M^2}\right)}$$

at $M = M_1$, $L_{max} = L_{max_1}$

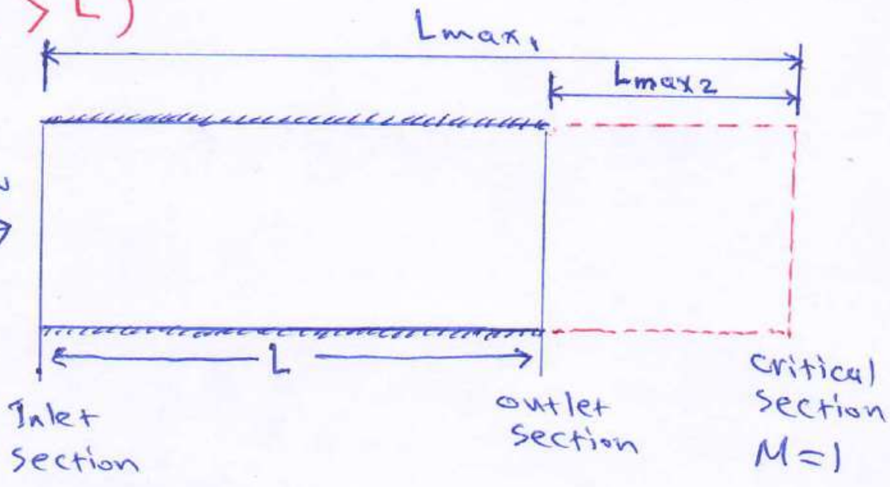
at $M = M_2$, $L_{max} = L_{max_2}$

L_{max} = Maximum possible length from the inlet section (L_{max_1}) or outlet section (L_{max_2}) to the critical section where $M = 1$

L = original duct length

There are two possible cases for L_{max_1}, L_{max_2} and L arrangement

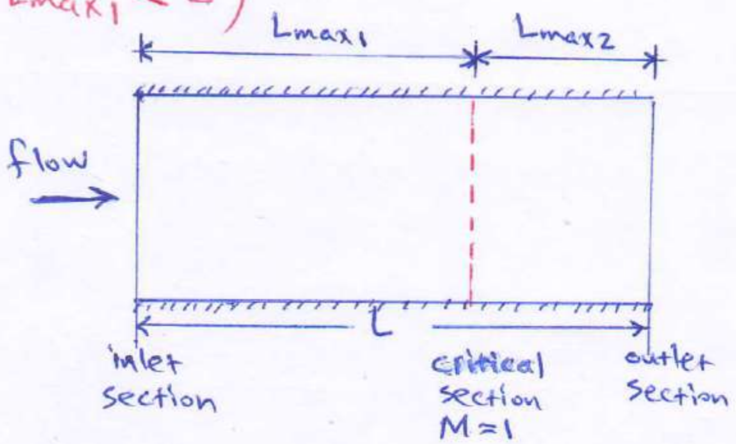
* first case $(L_{max_1} > L)$



then the relation between lengths is

$$\boxed{L_{max_2} = L_{max_1} - L} \quad \text{or} \quad \left.\frac{4f L_{max_2}}{D}\right|_{M_2} = \left.\frac{4f L_{max_1}}{D}\right|_{M_1} - \frac{4fL}{D}$$

* Second case $(L_{max_1} < L)$



for this case

$$\boxed{L_{max_2} = L - L_{max_1}} \quad \text{or} \quad \left.\frac{4f L_{max_2}}{D}\right|_{M_2} = \frac{4fL}{D} - \left.\frac{4f L_{max_1}}{D}\right|_{M_1}$$

* The Dimensionless Fanno Flow Properties (relative to the critical state)

$$\boxed{\frac{T}{T^*} = \frac{c^2}{c^{*2}} = \frac{\frac{\gamma+1}{2}}{1 + \frac{\gamma-1}{2} M^2}}$$

$$\boxed{\frac{P}{P^*} = \frac{1}{M} \sqrt{\frac{\frac{\gamma+1}{2}}{\left(1+\frac{\gamma-1}{2} \cdot M^2\right)}}}$$

$$\boxed{\frac{u}{u^*} = M \sqrt{\frac{\frac{\gamma+1}{2}}{\left(1+\frac{\gamma-1}{2} \cdot M^2\right)}}}$$

$$\boxed{\frac{\rho}{\rho^*} = \frac{u^*}{u} = \frac{1}{M} \cdot \sqrt{\left(\frac{2}{\gamma+1}\right) \cdot \left(1+\frac{\gamma-1}{2} \cdot M^2\right)}}$$

$$\boxed{\frac{P_o}{P_o^*} = \frac{1}{M} \left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2} M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$\boxed{\frac{F}{F^*} = \frac{1+\gamma M^2}{M\sqrt{2(\gamma+1)\left(1+\frac{\gamma-1}{2} M^2\right)}}}$$

, F = impulse function

$F = P \cdot A\left[1+\gamma M^2\right]$

$$\boxed{\frac{S-S^*}{R} = -\ln \frac{P_o}{P_o^*} = -\ln\left[\frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma+1}{2} M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right]}$$

Flow property	Flow Type	
	Subsonic flow	Supersonic flow
M	increase	decrease
u	〃	〃
P	decrease	increase
T	〃	〃
ρ	〃	〃

Ex.(6.1)

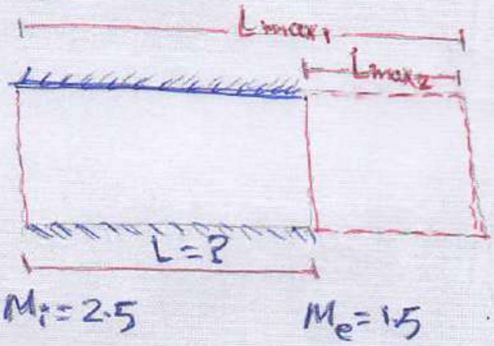
«Fanno Flow»

Air flow in a 5 cm diameter pipe. Air enters at $M=2.5$ and leaves at $M=1.5$, what is the pipe length required?. What would be the pipe length required to bring the flow to Mach number $M=1$ at exit?. Assume that $f=0.002$.

Solu.

$D=0.05\,m$

$f=0.002$



* من جدول جريان فانو وعند $M_i=2.5$

$$\frac{4fL_{max_1}}{D}=0.432=\frac{4\times0.002\times L_{max_1}}{0.05}$$

$$\Rightarrow L_{max_1}=2.7\ m$$

* من جدول جريان فانو وعند $M_e=1.5$

$$\frac{4fL_{max_2}}{D}=0.136=\frac{4\times0.002\times L_{max_2}}{0.05}$$

$$\Rightarrow \boxed{L_{max_2}=0.85\ m}$$

$$L=L_{max_1}-L_{max_2}=2.7-0.85$$

طول الأنبوب الفعلي $\boxed{L=1.85\ m}$

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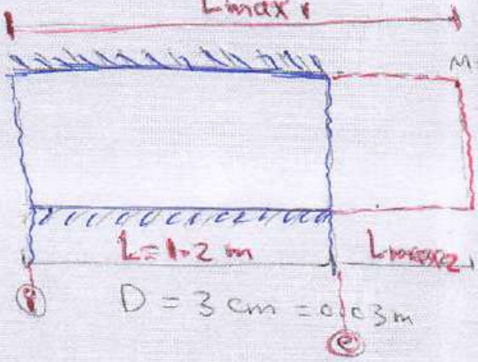
Ex-(6.2)

((Fanno Flow))

Air enters a constant area insulated duct of Length $L=1.2\,m$ and diameter $D=3\ cm$. Flow conditions at inlet are, Mach number $M=0.5$, a Pressure $P=120\ kpa$, and temperature $T=290\ k$. Assuming that the average friction coefficient $f=0.005$, determine:

- Mach number, Pressure and temperature at exit
- Stagnation pressure



$M_i = 0.5$, $P_i = 120\ kpa$, $T_i = 290\ K$, $f = 0.005$

Find: M_e , P_e , T_e , P_{oi}

Solu:

* من جدول الجريان الايزنتروبي ص 3 او الملحقات وعند $M_i = 0.5$

$$\frac{P_{oi}}{P_i} = \frac{1}{0.843} \Rightarrow \boxed{P_{oi} = 142.349\ kpa}$$

* من جدول جريان فانو وعند $M_i = 0.5$

$$\frac{4fL_{max_1}}{D} = 1.06906\ ,\ \frac{P_i}{P^*} = 2.13809\ ,\ \frac{T_i}{T^*} = 1.14286\ ,\ \frac{P_{oi}}{P_{oi}^*} = 1.34$$

$$\frac{4fL}{D} = 0.8 \Rightarrow L_{max_1} > L \Rightarrow L_{max_2} = L_{max_1} - L$$

$$\frac{4fL_{max_2}}{D} = 1.06906 - 0.8 = 0.2691$$

* باستخدام جدول فانو ومن معرفة $\frac{4fL_{max2}}{D}$ نجد قيمة M_e المقابلة

$$\Rightarrow \boxed{M_e = 0.672}\ ,\ \frac{P_e}{P^*} = 1.561\ ,\ \frac{T_e}{T^*} = 1.101\ ,\ \frac{P_{oe}}{P_{oe}^*} = 1.116$$

$$P_e = P_i\,\frac{P_e/P^*}{P_i/P^*} = 120 * \frac{1.561}{2.138} \Rightarrow \boxed{P_e = 87.623\ kpa}$$

$$T_e = T_i\,\frac{T_e/T^*}{T_i/T^*} = 290 * \frac{1.101}{1.143} \Rightarrow \boxed{T_e = 279.3\ K}$$

Mohammed Gh.Jehad

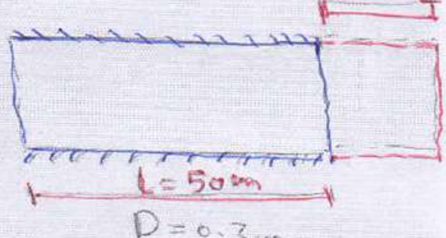
$E_x(6\text{-}3)$

«Fanno Flow»

Air flow out of a duct at a rate of 1000 m³/min. The pressure and temperature of air at exit are 150 kpa and 290 K respectively. The pipe has 50 m long, 30 cm diameter, and 0.005 friction factor. Find:

- Mach number at inlet and exit
- The pressure and temperature at inlet.

$\dot{Q} = 1000 \frac{m^3}{min} = \frac{1000}{60} = 16.67\ m^3/s$



$D = 0.3m$

$f = 0.005$

$P_e = 150\ kpa \quad , T_e = 290\ K$

$A = \pi D^2/4 = \pi\, 0.3^2/4 = 0.071\ m^2$

$U_e = \dot{Q}/A \Rightarrow U_e = 235.8\ \frac{m}{s}$

$C_e = \sqrt{\gamma R T_e} \Rightarrow C_e = 345.5\ \frac{m}{s}, \; M_e = \frac{U_e}{C_e} \Rightarrow \boxed{M_e = 0.683}$

* من جدول الجريان الايزنتروبي وعند $M_e = 0.683$

$\frac{P_{o_e}}{P_e} \approx \frac{1}{0.73} \Rightarrow P_{o_e} = 204.882\ kpa$

* من جدول جريان فانو وعند $M_e = 0.683$

$\frac{4f L_{max_2}}{D} = 0.244, \; \frac{P_e}{P^*} = 1.535, \; \frac{T_e}{T^*} = 1.098, \; \frac{P_{o_e}}{P_o^*} = 1.108$

$\frac{4fL}{D} = 3.333 \Rightarrow L_{max_2} = 3.645 \; , \; \frac{4f L_{max_1}}{D} = 3.5763$

$\Rightarrow \boxed{M_i \simeq 0.345}$

* من جدول فانو وعند $M_i = 0.345$

$\frac{P_i}{P^*} = 3.13874, \; \frac{T_i}{T^*} = 1.1721,$

$P_i = P_e \frac{P_i/P^*}{P_e/P^*} = \frac{3.13874}{1.535} * 150 \Rightarrow \boxed{P_i = 306.72\ kpa}$

$T_i = T_e \frac{T_i/T^*}{T_e/T^*} = \frac{1.1721}{1.098} * 290 \Rightarrow \boxed{T_i = 309.6\ K}$

Mohammed Gh. Jehad

Flow in Constant Area Ducts with Heat Transfer

" Rayleigh Flow"

The most common factors that affect the flow charactristics of compressible fluid flow are area change, wall friction and heat transfer. Compressible fluid flows with area change and wall friction have been disscussed in previous chapter, the flow of compressible fluid flow with heat addition or removal in constant area ducts is going to be considered.

Heat addition or removal may be as a result of heat transfer through the wall, or as a result of chemical reaction such as fuel combustion in combustion chambers, or the evaporation of liquid droplets or condensation water vapor in gas stream.

Governing Equations:

Consider the steady one-dimensional frictionless flow through a control volume element with length of "dx" for constant area duct shown in figure (7-1), where the heat is added or removed from the element at a rate of "dQ".

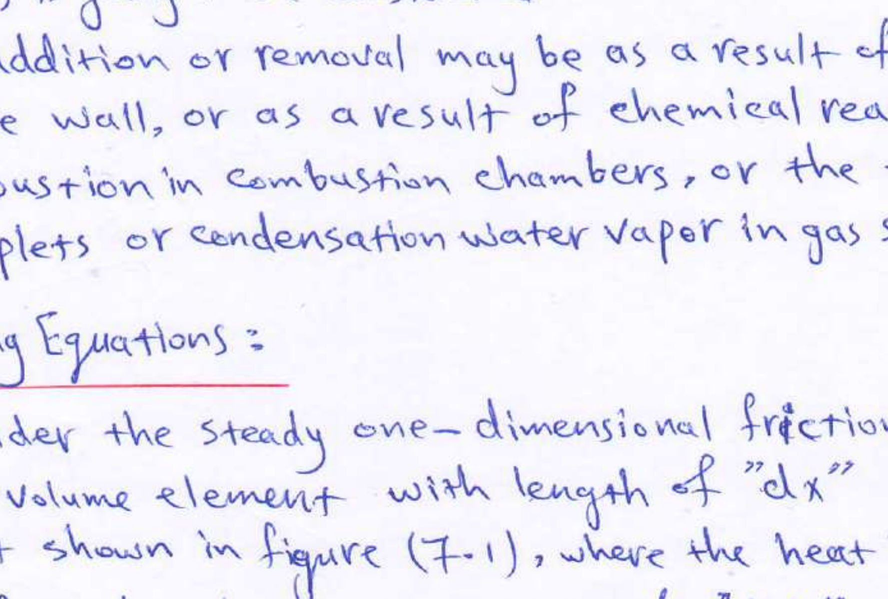


Figure (7-1)
Control Volume for the flow through constant area duct with heat transfer

* Continuity equation:

Since the flow is a constant area one, then the mass flow rate per unit area is constant.

$$\dot{m} = \rho u A = (\rho + d\rho)(u + du)\,A$$

then, $$\frac{d\rho}{\rho} + \frac{du}{u} = 0$$

* Momentum equation

Pressure force acting on the element is equal to the net change of the momentum of the flow.

$$PA - (P + dP)\cdot A = (\rho + d\rho)(u + du)^2 \cdot A - \rho u^2 \cdot A$$

neglecting the second order differential terms

$$\Rightarrow \; -dP \cdot A = \rho \cdot u \cdot A \cdot du$$

$$\Rightarrow \; \frac{dP}{P} + \gamma \cdot M^2 \cdot \frac{du}{u} = 0$$

* Energy equation:

Since there is heat transfer across the boundaries of the element "dx", then, the steady flow energy equation is:

$$dQ = (h + dh) + (u + du)^2/2 - h - (u^2/2)$$

$$dQ = dh_o = dh + d(u^2)/2$$

for perfect gases,

$$h = c_p \cdot T \quad , \quad c_p = \frac{\gamma}{\gamma - 1} R$$

then $$\frac{dT_o}{T_o} \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right) = \frac{dT}{T} + (\gamma - 1) M^2 \cdot \frac{du}{u}$$

where,

$$dQ = c_p \cdot dT_o$$

Equation of state:

For perfect gases, the equation of state is given by;

$$\frac{P}{\rho R T} = \text{Constant}$$

Then after differentiation, we get the following:

$$-\frac{dP}{P} + \frac{dT}{T} + \frac{d\rho}{\rho} = 0$$

Mach number definition:

$$M^2 = U^2 / C^2 = U^2 / \gamma R T$$

after differentiation, we get the following:

$$2\frac{dU}{U} - \frac{dT}{T} = \frac{dM^2}{M^2}$$

Rayleigh Flow working Formula and Relations:

The maximum stagnation temperature ratio $\left(\frac{T_o}{T_o^*}\right)$

$$\frac{T_o}{T_o^*} = \frac{2(\gamma+1) M^2 \left(1 + \frac{\gamma-1}{2} M^2\right)}{(1+\gamma M^2)^2}$$

$$Q = c_p (T_{o_2} - T_{o_1})$$

$$Q_{max} = c_p (T_o^* - T_o)$$

T_o^* = the maximum allowable stagnation temperature

The Dimensionless Rayleigh Flow Properties (Relative to the critical state)

$$\boxed{\frac{T}{T^*} = \frac{c^2}{c^{*2}} = \frac{(\gamma+1)^2 M^2}{(1+\gamma M^2)^2}}$$

$$\boxed{\frac{P}{P^*} = \frac{1+\gamma}{1+\gamma M^2}}$$

$$\boxed{\frac{u}{u^*} = \frac{(1+\gamma) M^2}{1+\gamma M^2}}$$

$$\boxed{\frac{\rho}{\rho^*} = \frac{1+\gamma M^2}{(1+\gamma) M^2}}$$

$$\boxed{\frac{P_o}{P_o^*} = \frac{1+\gamma}{1+\gamma M^2}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2} M^2\right)\right]^{\frac{\gamma}{\gamma-1}}}$$

$$\boxed{\begin{aligned}\frac{S-S^*}{R} &= -\ln\left[\left(\frac{T}{T^*}\right)^{\frac{\gamma}{\gamma-1}} \Big/ \left(\frac{P}{P^*}\right)\right] \\ &= -\ln\left[\left(\frac{1}{M}\right)^{\frac{2\gamma}{\gamma-1}} \left(\frac{1+\gamma M^2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]\end{aligned}}$$

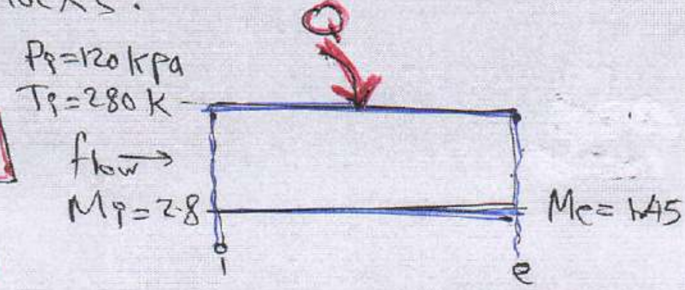
$$\frac{T_2}{T_1} = \frac{T_2}{T^*} * \frac{T^*}{T_1} = \frac{\left[\frac{T_2}{T^*}\right]_{M_2}}{\left[\frac{T_1}{T^*}\right]_{M_1}}$$

Heat transfer effects upon Rayleigh flow conditions

Flow Property	Flow type: Heating, Subsonic $M<1$	Heating, Supersonic $M>1$	Cooling, Subsonic $M<1$	Cooling, Supersonic $M>1$
Mach number, M	increases	decreases	decreases	increases
Velocity, u	increases	decreses	decreases	increases
Pressure, P	decreases	increases	increases	decreases
Temperature T	increases $M<1/\sqrt{\gamma}$ decreases $M>1/\sqrt{\gamma}$	increases	decreases $M<1/\sqrt{\gamma}$ increases $M>1/\sqrt{\gamma}$	decreases
Density, ρ	decreases	increases	increases	decreases
P_o	decreases	decreases	increases	increases
T_o	increases	increases	decreases	decreases
Entropy, S	increases	increases	decreases	decreases

Ex.:(7.1) « Rayleigh Flow »

Air is flowing through constant area duct with initial Conditions of 120 kpa pressure, 280 K temperature and 2.8 Mach number. Heat is transferred to the air as it flows through the duct such the Mach number at exit is 1.45. Assuming that frictional effect is nigligible, Find the pressure, temperature, stagnation pressure and temperature and the amount of heat transferred, and the exit conditions, if the flow is free of shocks.



* من جدول الجريان الايزنتروبي وعند $M_i = 2.8$

$$P_{oi} = P_i * \frac{P_{oi}}{P_i} \Rightarrow \boxed{P_{oi} = 3256.6\ kpa}$$

$$T_{oi} = T_i * \frac{T_{oi}}{T_i} \Rightarrow \boxed{T_{oi} = 719\ k}$$

* من جداول جريان رايليه وعند $M_i = 2.8$

$$\frac{T_{oi}}{T_o^*} = 0.6738, \quad \frac{T_i}{T^*} = 0.3149$$

$$\frac{P_{oi}}{P_o^*} = 2.8731, \quad \frac{P_i}{P^*} = 0.2004, \quad \frac{\rho_i}{\rho^*} = 0.6365$$

* من جداول جريان رايليه وعند $M_e = 1.45$

$$\frac{T_{oe}}{T_o^*} = 0.9218, \quad \frac{P_{oe}}{P_o^*} = 1.0983$$

$$\frac{T_e}{T^*} = 0.7787, \quad \frac{P_e}{P^*} = 0.6086, \quad \frac{\rho_e}{\rho^*} = 0.7815$$

$$\Rightarrow T_{oe} = T_{oi} * \frac{T_{oe}/T_o^*}{T_{oi}/T_o^*} = \frac{0.9218}{0.6738} * 719 \Rightarrow \boxed{T_{oe} = 983.64\ K}$$

$$\Rightarrow P_{oe} = P_{oi} * \frac{P_{oe}/P_o^*}{P_{oi}/P_o^*} = \frac{1.0983}{2.8731} * 3256.6 \Rightarrow \boxed{P_{oe} = 1244.9\ kpa}$$

$$\Rightarrow T_e = T_i * \frac{T_e/T^*}{T_i/T^*} = \frac{0.7787}{0.3149} * 280 \Rightarrow \boxed{T_e = 692.4}$$

$$P_e = P_i * \frac{P_e/P^*}{P_i/P^*} = \frac{0.6086}{0.2004} * 120 \Rightarrow \boxed{P_e = 364.43\ kpa}$$

* لحساب كمية الحرارة يستخدم القانون

حفظ $\boxed{Q = C_p (T_{oe} - T_{oi})}$

حفظ $\boxed{C_{p_{air}} = 1.005\ kJ/kg.k}$

$$Q = 1.005 (983.64 - 719)$$

$$\Rightarrow \boxed{Q = 265.96\ kJ/kg}$$

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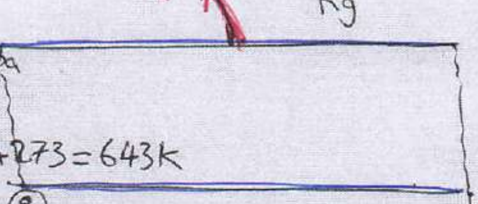
Ex (7-2) « Rayleigh Flow »

Air flows through constant area duct. At inlet section, the Pressure, temperature and Mach number are 250 kpa, 370°C and 0.6 respectively. Heat to be transferred from the air at a rate of 450 $\frac{kJ}{kg}$. Assuming that friction is negligible, find out the Mach number, Pressure, temperature, Stagnation pressure and temperature at exit of the pipe.

$-Q = 450 \frac{kJ}{kg}$



$M_i = 0.6$ * من جدول الجريان الايزنتروبي وعند

$P_{oi} = \frac{P_{oi}}{P_i} * P_i \Rightarrow \boxed{P_{oi} = 318.88\,kPa}$

$T_{oi} = \frac{T_{oi}}{T_i} * T_i \Rightarrow \boxed{T_{oi} = 689.3\,K}$

$M_i = 0.6$ * من جداول جريان رالية وعند

$\frac{T_{oi}}{T_o^*} = 0.8189$, $\frac{T_i}{T^*} = 0.9167$

$\frac{P_{oi}}{P_o^*} = 1.0753$, $\frac{P_i}{P^*} = 1.5957$, $\frac{\rho_i}{\rho^*} = 1.7407$

$Q = c_p (T_{oe} - T_{oi}) \Rightarrow -450 = 1.005 (T_{oe} - 689.3)$

$\Rightarrow \boxed{T_{oe} = 241.54\,K} \Rightarrow \frac{T_{oe}}{T_o^*} = \frac{T_{oe}}{T_{oi}} * \frac{T_{oi}}{T_o^*} = \frac{241.54}{689.3} * 0.8189 \Rightarrow \boxed{\frac{T_{oe}}{T_o^*} = 0.287}$

* من معرفة قيمة (T_{oe}/T_o^*) ومن جداول رالية نجد قيمة M_e المقابلة

$\Rightarrow M_e = 0.267$

$\frac{T_{oe}}{T_o^*} = 0.2867$, $\frac{T_e}{T^*} = 0.3392$, $\frac{P_{oe}}{P_o^*} = 1.2114$, $\frac{P_e}{P^*} = 2.1824$, $\frac{\rho_e}{\rho^*} = 6.4338$

$\Rightarrow T_{oe} = T_{oi} * \frac{T_{oe}/T_o^*}{T_{oi}/T_o^*} = 689.3 * \frac{0.2867}{0.8189} \Rightarrow \boxed{T_{oe} = 241.33\,k}$

$\Rightarrow P_{oe} = P_{oi} * \frac{P_{oe}/P_o^*}{P_{oi}/P_o^*} = 318.88 * \frac{1.2114}{1.0753} \Rightarrow \boxed{P_{oe} = 359.24\,kpa}$

$\Rightarrow T_e = T_i * \frac{T_e/T^*}{T_i/T^*} = 643 * \frac{0.3392}{0.9167} \Rightarrow \boxed{T_e = 237.92\,k}$

$\Rightarrow P_e = P_i \frac{P_e/P^*}{P_i/P^*} = 250 * \frac{2.1824}{1.5957} \Rightarrow \boxed{P_e = 341.92\,kpa}$

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